Due Sun

2.3 – Properties of Determinants; Cramer's Rule

Properties of Determinants

For A an $n \times n$ matrix

- $\det(kA) = k^n \det(A)$

one factor of k for each row of A

Reminder:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow \det(A) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$$

$$\begin{vmatrix} 3 & 6 \\ 3 & 4 \end{vmatrix} = -6 = 3(-2) = 3\det(A) \quad (2.2)$$

$$\text{But } 3A = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} \Rightarrow \det(3A) = \begin{vmatrix} 3 & 6 \\ 9 & 12 \end{vmatrix}$$

$$= -18 = 3^2(-2)$$

$$3\det A = 3\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \quad \text{but } 3A = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

- **[Lemma 2.3.2** (prelude to a later result)
  If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then $\det(EB) = \det(E)\det(B)$.] (In brackets because it's here to support the next point)

Suppose E results from multiplying the i^{th} row of I by k.

Then EB is

$$EB = \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vdots \\ k\vec{e}_i \\ \vdots \\ \vec{e}_n \end{bmatrix} B = \begin{bmatrix} \vec{e}_1 B \\ \vec{e}_2 B \\ \vdots \\ k\vec{e}_i B \\ \vdots \\ \vec{e}_n B \end{bmatrix}$$

So the ith row of EB is the ith row of B, multiplied by k.

$\det(E) = k$, so $\det(EB) = k \det(B)$
$= \det(E)\det(B)$.

- **Theorem 2.3.4** If A and B are square matrices of the same size, then $\det(AB) = \det(A)\det(B)$.

outline of proof.

If A and/or B is not invertible, then AB is not invertible (see Thm 2.3.3 below) and $0 = 0$. If A & B are invertible, we use products of elementary matrices and Lemma 2.3.2.

- **Theorem 2.3.5** If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

Pf: A invertible $\Rightarrow$ there is A^{-1} such that

$$AA^{-1} = I$$

$$\Rightarrow \det(AA^{-1}) = \det(I) = 1$$

$$\Rightarrow \det(A)\det(A^{-1}) = 1 \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$

because $\det(A) \neq 0$. ✓

Theorem 2.3.3 A square matrix A is invertible if and only if $\det(A) \neq 0$.

Theorem 2.3.8 Equivalent Statements (extends Theorem 1.6.4)

If A is an $n \times n$ matrix, then the following statements are equivalent.

a) A is invertible.

b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

c) The reduced row echelon form of A is I_n.

d) A is expressible as a product of elementary matrices.

e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

g) $\det(A) \neq 0$.

Pf: $\Rightarrow$ A invertible $\Rightarrow$ A is a product of elementary matrices $\Rightarrow$ $\det(A) \neq 0$.

$\Leftarrow$ $\det(A) \neq 0 \Rightarrow \text{rref}(A)$ does not have a row of zeros $\Rightarrow \text{rref}(A) = I$

$\Rightarrow$ A is invertible ✓

16. Find the values of k for which the matrix A is invertible.

$A = \begin{bmatrix} k & 2 \\ 2 & k \end{bmatrix}$ Not invertible if $\det(A) = 0 \Rightarrow \begin{vmatrix} k & 2 \\ 2 & k \end{vmatrix} = 0$

$k^2 - 4 = 0 \Rightarrow k = \pm 2$

A is invertible for all $k \neq \pm 2$.

35. In each part, find the determinant given that A is a 3×3 matrix for which $\det(A) = 7$.

a. $\det(3A)$

b. $\det(A^{-1})$

c. $\det(2A^{-1})$

d. $\det((2A)^{-1})$

a) $3^3 \cdot 7 = 189$

b) $\frac{1}{\det(A)} = \frac{1}{7}$

c) $2^3\left(\frac{1}{7}\right) = \frac{8}{7}$

d) $\frac{1}{2^3 \cdot 7} = \frac{1}{56}$

Definition 1: If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the **matrix of cofactors from A**. The transpose of this matrix is called the **adjoint of A** and is denoted by $\text{adj}(A)$.

Theorem 2.3.6 Inverse of a Matrix Using Its Adjoint

If A is an invertible matrix, then $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$.

$\Rightarrow \det(A) \neq 0$

20. Decide whether the matrix is invertible, and if so, use the adjoint to find its inverse.

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$$

$$\det(A) = 2(-12) + 0(-4) + 3(6)$$
$$= -6 \neq 0 \quad A \text{ is invertible}$$

$M_{11} = \begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix} = -12$ $\quad C_{11} = -12$

$M_{12} = \begin{vmatrix} 0 & 2 \\ -2 & -4 \end{vmatrix} = 4$ $\quad C_{12} = -4$

$M_{13} = \begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix} = 6$ $\quad C_{13} = 6$

$M_{21} = 0$ $\quad C_{21} = 0$

$M_{22} = -2$ $\quad C_{22} \; -2$

$M_{23} = 0$ $\quad C_{23} = 0$

$M_{31} = -9$ $\quad C_{31} = -9$

$M_{32} = 4$ $\quad C_{32} = -4$

$M_{33} = 6$ $\quad C_{33} = 6$

$$[C_{ij}] = \begin{bmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{bmatrix} \qquad \operatorname{adj}(A) = [C_{ij}]^T = \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = -\frac{1}{6} \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & 0 & 3/2 \\ 2/3 & 1/3 & 2/3 \\ -1 & 0 & -1 \end{bmatrix}$$

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow [C_{ij}]=\begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \Rightarrow \operatorname{adj}(A)=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\Rightarrow A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem 2.3.7 Cramer's Rule

If $A\mathbf{x}=\mathbf{b}$ is a system of n linear equations in n unknowns such that $\det(A)\neq 0$, then the system has a unique solution. This solution is

$$x_1=\frac{\det(A_1)}{\det(A)}, \qquad x_2=\frac{\det(A_2)}{\det(A)}, \dots, \qquad x_n=\frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix

$$\mathbf{b}=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

26. Solve by Cramer's rule.

$$\begin{aligned} x-4y+z&=6 \\ 4x-y+2z&=-1 \\ 2x+2y-3z&=-20 \end{aligned}$$

$$A=\begin{bmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$-2+4+48=50$$

$$\det(A)=\begin{vmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix} \begin{matrix} 1 & -4 \\ 4 & -1 \\ 2 & 2 \end{matrix} \qquad -5-50=-55$$

$$3-16+8=-5$$

$$\det(A_1)=\begin{vmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{vmatrix}=144$$

$$\det(A_2)=\begin{vmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{vmatrix}=61$$

$$\det(A_3) = \begin{vmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{vmatrix} = -230$$

$$x = \frac{\det(A_1)}{\det(A)} = -\frac{144}{55}$$

$$y = \frac{\det(A_2)}{\det(A)} = -\frac{61}{55}$$

$$z = \frac{\det(A_3)}{\det(A)} = \frac{230}{55} = \frac{46}{11}$$

$$(x, y, z) = \left(-\frac{144}{55}, -\frac{61}{55}, \frac{46}{11}\right)$$